

VERONESE VARIETIES CONTAINED IN HYPERSURFACES

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ABSTRACT. Alex Waldron proved that for sufficiently general degree d hypersurfaces in projective n -space, the Fano scheme parameterizing r -dimensional linear spaces contained in the hypersurface is nonempty precisely for the degree range $n \geq N_1(r, d)$ where the “expected dimension” $f_1(n, r, d)$ is nonnegative, in which case $f_1(n, r, d)$ equals the (pure) dimension. Using work by Gleb Nenashev, we prove that for sufficiently general degree d hypersurfaces in projective n -space, the parameter space of r -dimensional e -uple Veronese varieties contained in the hypersurface is nonempty of pure dimension equal to the “expected dimension” $f_e(n, r, d)$ in a degree range $n \geq \tilde{N}_e(r, d)$ that is asymptotically sharp. Moreover, we show that for $n \geq 1 + N_1(r, d)$, the Fano scheme parameterizing r -dimensional linear spaces is irreducible.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let k be an algebraically closed field, not necessarily of characteristic 0. For integers $n, r, e > 0$, a *Veronese e -uple r -fold* in \mathbb{P}_k^n is the image of a morphism $\nu : \mathbb{P}_k^r \rightarrow \mathbb{P}_k^n$ such that $\nu^*\mathcal{O}(1)$ is isomorphic to $\mathcal{O}(e)$ and such that the pullback homomorphism,

$$\nu_1^* : H^0(\mathbb{P}_k^n, \mathcal{O}(1)) \rightarrow H^0(\mathbb{P}_k^r, \mathcal{O}(e)),$$

is surjective; such a morphism is a closed immersion. For brevity, the image $\nu(\mathbb{P}^r)$ of such a morphism is called a V_e^r . Denoting by $P_r(t) \in \mathbb{Q}[t]$ the numerical polynomial with $P_r(d) = \binom{d+r}{r}$ for all integers $d \geq -r$, the $\mathcal{O}(1)$ -Hilbert polynomial of the image of ν equals $P_r(et)$. Denote $P_r(e) - 1$ by $n_e(r)$, e.g., $n_1(r)$ equals r , and $n_2(r)$ equals $r(r+3)/2$. In the Hilbert scheme $\text{Hilb}_{\mathbb{P}_k^n/k}^{P_r(et)}$ there is an open subscheme $G_e(r, \mathbb{P}_k^n)$ parameterizing Veronese e -uple r -folds. This open scheme is nonempty precisely when $h^0(\mathbb{P}_k^n, \mathcal{O}(1)) \geq h^0(\mathbb{P}^r, \mathcal{O}(e))$, i.e., when $n \geq n_e(r)$. When it is nonempty, $G_e(r, \mathbb{P}_k^n)$ is smooth and geometrically integral of dimension

$$f_e(n, r) := (n+1)(n_e(r)+1) - (r+1)^2.$$

Note, in particular, that $f_1(n, r)$ equals $(n-r)(r+1)$, which is nonnegative if and only if $n \geq r$, i.e., $n \geq n_1(r)$. Please note, for every $e \geq 2$, for every $n \geq 1$, for every $r \geq 1$, $f_e(n, r)$ is positive, even though $G_e(r, \mathbb{P}_k^n)$ is empty for $n < n_e(r)$. In fact, $G_e(r, \mathbb{P}_k^n)$ has a natural action of \mathbf{PGL}_{n+1} under which it is smoothly homogeneous (the stabilizer subgroup is reduced). Assuming that it is nonempty, the quasi-projective scheme $G_e(r, \mathbb{P}_k^n)$ is projective precisely when r equals 1. For $r = 1$, this is a classical Grassmannian, $G_1(r, \mathbb{P}_k^n) = \text{Grass}(r, \mathbb{P}_k^n)$, parameterizing r -dimensional projective linear subspaces of \mathbb{P}_k^n .

For every (locally) closed subscheme $X \subset \mathbb{P}_k^n$, the *Fano scheme* of Veronese e -uple r -folds in X is the intersection $F_e(r, X)$ of the open subscheme $G_e(r, \mathbb{P}_k^n)$ with the

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(locally) closed subscheme $\text{Hilb}_{X/k}^{P_r(et)}$ in $\text{Hilb}_{\mathbb{P}^n/k}^{P_r(et)}$. Even though the V_e^r varieties are among the simplest r -cycles in \mathbb{P}_k^n , little is known about $F_e(r, X)$ for general X . Here we study $F_e(r, X)$ for degree d hypersurfaces X in \mathbb{P}_k^n . In that case, there is a lower bound on the dimension of every (nonempty) irreducible component as follows,

$$f_e(n, r, d) = f_e(n, r) - P_r(ed) = (n+1)P_r(e) - P_r(de) - (r+1)^2.$$

Denote by $N_e(r, d)$ the smallest integer $n \geq n_r(e)$ such that $f_e(n, r, d) \geq 0$, i.e.,

$$N_e(r, d) = \max \left(-1 + P_r(e), -1 + \left\lceil \frac{(r+1)^2 + P_r(ed)}{P_r(e)} \right\rceil \right).$$

It is occasionally convenient to use $M_e(r, d) = N_e(r, d) - P_r(e) + 1$, i.e.,

$$M_e(r, d) = \max \left(0, \left\lceil \frac{(r+1)^2 + P_r(ed) - P_r(e)^2}{P_r(e)} \right\rceil \right).$$

Recall that the complete linear system $\mathbb{P}H^0(\mathbb{P}_k^n, \mathcal{O}(d))$ parameterizing degree d hypersurfaces in \mathbb{P}_k^n is canonically k -isomorphic to the Hilbert scheme $\text{Hilb}_{\mathbb{P}^n/k}^{P_n(t)-P_n(t-d)}$ of closed subschemes of \mathbb{P}_k^n . Denote by $\mathcal{X} \subset \mathbb{P}H^0(\mathbb{P}_k^n, \mathcal{O}(d)) \times_{\text{Spec}(k)} \mathbb{P}_k^n$ the universal closed subscheme such that the restricted projection k -morphism is flat,

$$\pi : \mathcal{X} \rightarrow \mathbb{P}H^0(\mathbb{P}_k^n, \mathcal{O}(d)).$$

Denote by $F_e(r, \mathcal{X}) \subset \mathbb{P}H^0(\mathbb{P}_k^n, \mathcal{O}(d)) \times_{\text{Spec}(k)} G_e(r, \mathbb{P}_k^n)$ the relative Fano scheme of Veronese e -uple r -folds in fibers of π , i.e., the intersection of the open subscheme $\mathbb{P}H^0(\mathbb{P}_k^n, \mathcal{O}(d)) \times_{\text{Spec}(k)} F_e(r, \mathbb{P}_k^n)$ with the relative Hilbert scheme $\text{Hilb}_{\pi}^{P_r(et)}$. The two projection morphisms restrict to morphisms on $F_e(r, \mathcal{X})$,

$$\rho : F_e(r, \mathcal{X}) \rightarrow G_e(r, \mathbb{P}_k^n),$$

$$\pi : F_e(r, \mathcal{X}) \rightarrow \mathbb{P}H^0(\mathbb{P}_k^n, \mathcal{O}(d)).$$

Denote by $F_e(r, \mathcal{X})_{\text{fl}}$, resp. $F_e(r, \mathcal{X})_{\text{sm}}$, the maximal open subscheme of $F_e(r, \mathcal{X})$ on which π is flat, resp. smooth. Denote by $U_{e,d}^r \subset \mathbb{P}H^0(\mathbb{P}_k^n, \mathcal{O}(d))$, resp. $V_{e,d}^r \subset U_{e,d}^r$, the maximal open subscheme over which π is flat, resp. smooth. The open $U_{e,d}^r$ is a dense open by the Generic Flatness Theorem.

Proposition 1.1. *For all $n \geq n_e(r)$, for all $d \geq 1$, the morphism ρ is a Zariski locally trivial projective bundle of relative dimension $P_n(d) - P_r(de)$. Thus the restriction of π over $U_{e,d}^r$ either has empty fibers or else it is flat of relative dimension $f_e(n, r, d)$. If the characteristic equals 0 or $p \geq p_e(n, r, d)$ for an effectively computable integer $p_e(n, r, d)$, then $V_{e,d}^r$ is a dense open subset of $U_{e,d}^r$, i.e., π is smooth over a dense open of $U_{e,d}^r$.*

By the proposition, in order that $F_e(r, X)$ is nonempty for every degree d hypersurface X , it is necessary that $n \geq n_r(e)$ and $f_e(n, r, d) \geq 0$. This is equivalent to the condition that $n \geq N_e(r, d)$. Also, for the difference $m := n - n_e(r)$, it is equivalent to the condition that $m \geq M_e(r, d)$. Using a theorem of Hochster-Laksov, Alex Waldron proved that the necessary condition is sufficient for $e = 1$, i.e., for linear spaces.

Theorem 1.2 (Waldron, [Wal08]). *For all $d \geq 3$, for all $n \geq N_1(r, d) = r + \lceil P_r(d)/(r+1) \rceil$, the smooth locus of π in $F_1(r, \mathcal{X})$ is dense, thus, for every degree d hypersurface X in \mathbb{P}_k^n , the Fano scheme $F_1(r, X)$ of linear r -planes in X is nonempty. For $d = 1$, this is true for all $n \geq N_1(r, 1) = 1 + n_1(r) = 1 + r$. For $d = 2$, this is true precisely for $n \geq 1 + 2r$.*

Please note, when $d = 2$, then $1 + 2r > N_1(r, 2)$ for all $r \geq 2$, so this case is special. Recently, Gleb Nenashev has generalized the Hochster-Laksov theorem. Using this generalization, there is a similar result for all $e \geq 2$ for a bound $n \geq \tilde{N}_e(r, d)$, where $\tilde{N}_e(r, d)$ is asymptotically sharp for fixed e, r , and increasing d .

Theorem 1.3. *For $e \geq 2$, for all n at least $\tilde{N}_e(r, d) := -1 + 2P_r(e) + \lceil P_r(de)/P_r(e) \rceil$, the smooth locus of π in $F_e(r, \mathcal{X})$ is dense. Thus, for every sufficiently general degree d hypersurface X in \mathbb{P}_k^n , the Fano scheme $F_e(r, X)$ of Veronese e -uple r -folds in X is a nonempty, geometrically reduced, local complete intersection scheme of dimension $f_e(r, d, n)$. For $d = 1$, this is true precisely for $n \geq 1 + n_e(r)$. For $d = 2$, this is true precisely for $n \geq n_e(r) = N_e(r, 2)$.*

What about irreducibility, i.e., connectedness? The method we use to study this, based on Minoccheri's form of Bertini's irreducibility theorem, cf. [Min16], uses projective parameter spaces. So the result works best for linear spaces, $e = 1$. The integer $N'_e(r, d)$ is the least integer n such that the complement of the smooth locus $F_e(r, \mathcal{X})_{\text{sm}}$ of π in $F_e(r, \mathcal{X})$ has codimension ≥ 2 everywhere.

Proposition 1.4. *If $n \geq N'_1(r, d)$, then for every degree d hypersurface X in \mathbb{P}_k^n , the Fano scheme $F_1(r, X)$ is geometrically connected. For sufficiently general X , the Fano scheme is a local complete intersection scheme that is geometrically integral and normal. If the characteristic is 0 or $p > p_1(n, r, d)$, then for sufficiently general X , the Fano scheme is also smooth.*

This connectedness result for linear spaces implies connectedness results for more general cycles. The following corollary is one example of this; certainly the bound can be improved.

Corollary 1.5. *If $n \geq N'_1(d, n_e(r))$, then there exists a dense, Zariski open subscheme $W_{e,d}^r \subset U_{e,d}^r$ such that for every degree d hypersurface X with $[X] \in W_{e,d}^r$, the Fano scheme $F_e(r, X)$ is geometrically connected. If the characteristic is 0 or $p > p_1(n, r, d)$, then for sufficiently general X , the Fano scheme is also smooth.*

The following bound for $N'_1(r, d)$ is sharp to within 1. There are infinitely many cases when the bound is sharp.

Theorem 1.6. *Regarding $N'_1(r, d)$, we have the following.*

- (i) *For all $d \geq 2$, $N_1(r, d) \leq N'_1(r, d) \leq 1 + N_1(r, d)$, so the Fano schemes $F_1(r, X)$ are geometrically connected if $n \geq 1 + N_1(r, d)$.*
- (ii) *For $d = 1$, $N_e(r, 1)$ equals $1 + n_e(r)$, and the Fano schemes $F_1(r, X)$ are connected for all $n \geq n_e(r)$.*
- (iii) *For $d = 2$, $N_1(r, 2)$ equals $2r + 1$, $N'_1(r, 2)$ equals $1 + N_1(r, 2) = 2r + 2$, and the Fano schemes $F_1(r, X)$ are geometrically disconnected for $n = N_1(r, 2) = 2r + 1$.*

- (iv) For all $d \geq 2$, if $f_1(N_1(r, d), r, d)$ equals 0, then $N'_1(r, d)$ equals $1 + N_1(r, d)$; in fact, the length of the finite Fano scheme for $n = N_1(r, d)$ is divisible by d^{r+1} .

The case of linear spaces deserves special attention. In characteristic 0, for $n \geq N'_1(r, d)$, for X sufficiently general, $F_1(r, X)$ is smooth and geometrically irreducible of the expected dimension. What can we say for *every* smooth hypersurface X ? In an appropriate degree range, there exists a canonically defined (nonempty) irreducible component of $F_1(r, X)$ of the expected dimension such that $F_1(r, X)$ is reduced at the generic point of this component. It is convenient to introduce the flag Fano scheme.

For every scheme S over $\text{Spec}(\mathbb{Q})$, for every \mathbb{P}^n -bundle $\pi : \mathbb{P}_S(E) \rightarrow S$ together with an ample invertible sheaf $q : \pi^* E^\vee \rightarrow \mathcal{O}_E(1)$, for every locally closed subscheme $X \subset \mathbb{P}_k^n$ such that $\pi : X \rightarrow S$ is locally finitely presented, denote by $F_1(0, 1, \dots, r, X/S)$ the flag Hilbert scheme of X , $\text{fHilb}_{X/S}^{P_0(t), P_1(t), \dots, P_r(t)}$, parameterizing flags of linear subspaces contained in fibers of X . There is a forgetful S -morphism,

$$\rho_r : F_1(0, 1, \dots, r-1, r, X/S) \rightarrow F_1(0, 1, \dots, r-1, X/S).$$

Assume now that X is S -smooth. Then for every component of $F_1(0, 1, \dots, r, X/S)$ parameterizing flags of linear subspace $\Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda_r$ in geometric fibers of X/S , there is a lower bound $e_r(X/S)$ on the dimension of every irreducible component of every (nonempty) fiber of ρ_r ,

$$e_r(X/S) = -r - 1 + \sum_{\ell=1}^r b_{r,\ell} \langle \text{ch}_\ell(T_{X/S}), [\Lambda_\ell] \rangle,$$

where $\text{ch}_\ell(T_{X/S})$ is the graded piece of the Chern character of homogeneous degree ℓ of $T_{X/S} = (\Omega_{X/S})^\vee$, and where the rational numbers $b_{r,\ell}$ are determined by

$$P_r(t-1) = \sum_{\ell=1}^r \frac{b_{r,\ell}}{\ell!} t^\ell.$$

There is a natural infinitesimal deformation theory and obstruction theory for ρ_r . When the obstruction group vanishes, then ρ_r is smooth of relative dimension $e_r(X/S)$. There exists a sequence $(U_\ell)_{0 \leq \ell \leq r-1}$ of open subschemes $U_\ell \subset F_1(0, 1, \dots, \ell, X/S)$ such that

- (i) $U_0 \subset X$ is the maximal open subscheme over which ρ_1 has vanishing obstruction groups so that ρ_1 is smooth of relative dimension $e_1(X/S)$ over U_0 ,
- (ii) for every $\ell = 1, \dots, r-1$, $U_\ell \subset \rho_\ell^{-1}(U_\ell)$ is the maximal open subscheme over which $\rho_{\ell+1}$ has vanishing obstruction groups so that $\rho_{\ell+1}$ is smooth of relative dimension $e_{\ell+1}(X/S)$ over U_ℓ .

This sequence is compatible with arbitrary base change over S . The main result of [Sta06] is the following.

Proposition 1.7. [Sta06] *Assume that k has characteristic 0. Let $X \subset \mathbb{P}_k^n$ be a smooth hypersurface of degree d .*

- (i) *If $n < r + P_r(d-1)$, then $\rho_r^{-1}(U_{r-1})$ is empty.*

- (ii) If $n \geq r + P_r(d - 1)$, then $\rho_r^{-1}(U_{r-1})$ is nonempty, the restriction of ρ_r over U_{r-1} is smooth and projective of relative dimension $n - r - P_r(d - 1)$, and each geometric fiber is a complete intersection in a projective space of a sequence of hypersurfaces whose maximal degree equals d .
- (iii) If n equals $r + P_r(d - 1)$ and $d > 1$, then the fibers of ρ_r over U_{r-1} are not geometrically connected.
- (iv) If $n \geq 1 + r + P_r(d - 1)$, then every geometric fiber of ρ_r over U_r is geometrically connected so that $\rho_r^{-1}(U_r)$ is smooth and irreducible.

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2. PROOF OF PROPOSITION 1.1

This proposition follows by the general method of incidence correspondences. Let S be a scheme. Let E be a locally free \mathcal{O}_S -module of rank $n + 1$. Let $\pi_0 : \mathbb{P}_S(E) \rightarrow S$ together with $\pi_0^* E^\vee \rightarrow \mathcal{O}_E(1)$ represent the functor that associates to every S -scheme, $f : T \rightarrow S$, the set of invertible sheaf quotients on T of $f^* E^\vee$. Then π is a \mathbb{P}^n -bundle over S .

For every locally closed subscheme $X \subset \mathbb{P}_S(E)$ that is locally finitely presented over S (automatic for Noetherian schemes), for every integer r , the associated *Fano scheme*, $F_1(r, X/S)$, is the Hilbert scheme $\text{Hilb}_{X/S}^{P_r(t)}$. Hilbert polynomials are with respect to the invertible sheaf $\mathcal{O}_E(1)$. Of course $F_1(r, \mathbb{P}(E)/S)$ is the Grassmannian bundle associated to E , i.e., an S -scheme $\pi_r : \text{Grass}_S(r + 1, E) \rightarrow S$ together with the locally free quotient $\pi_r^* E^\vee \rightarrow Q_{E,r}$ of rank $r + 1$ that represents the functor sending $f : T \rightarrow S$ to the set of rank $r + 1$ locally free quotients of $f^* E^\vee$.

For every integer $d \geq 0$, denote by $S_d(E)$ the locally free sheaf $(\pi_0)_* \mathcal{O}_E(d)$, so that the direct sum $S(E) := \bigoplus_{d \geq 0} S_d(E)$ with its natural product structure is the homogeneous coordinate ring of $\mathbb{P}_S(E)$ with respect to $\mathcal{O}_E(1)$. In other words, $S_d(E)$ is the degree d symmetric power of E^\vee , i.e., for the tensor algebra $T(E)$ of E^\vee , the algebra quotient $T(E) \rightarrow S(E)$ is initial among morphisms of sheaves of associative \mathcal{O}_S -algebras that are commutative. Since the tensor algebra of the invertible sheaf $\mathcal{O}_E(1)$ on $\mathbb{P}_S(E)$ is commutative, the invertible quotient $\pi_0^* S_1(E) \rightarrow \mathcal{O}_V(1)$ induces an invertible quotient $\pi_0^* S_d(E) \rightarrow \mathcal{O}_V(d)$. Thus, on the fiber product $\mathbb{P}_S(S_d(E)) \times_S \mathbb{P}_S(E)$, there is a natural morphism of invertible sheaves,

$$\alpha : \text{pr}_1^* \mathcal{O}_{S_d(E)}(-1) \rightarrow \text{pr}_2^* \mathcal{O}_E(d).$$

The support of the cokernel of α is a Cartier divisor $\mathcal{X} \subset \mathbb{P}_S(S_d(E)) \times_S \mathbb{P}_S(E)$ that is flat with respect to pr_1 and has relative degree d with respect to $\text{pr}_2^* \mathcal{O}_E(1)$.

For every separated, finitely presented morphism, $\pi : Z \rightarrow S$, for every quasi-coherent \mathcal{O}_Z -module \mathcal{E} that is locally finitely presented, that is \mathcal{O}_S -flat, and that has proper support over S , there is a maximal open subscheme $U = U_{\pi, \mathcal{E}} \subset S$ such that the complement of U equals the (locally finite) union of the supports of $R^q \pi_* \mathcal{E}$ for the (locally finitely) many $q > 0$ such that $R^q \pi_* \mathcal{E}$ is nonzero. By [Gro63, Corollaire 7.9.10, Lemme 7.9.10.1], for every S -scheme $f : T \rightarrow S$, for the base change $\pi_T : Z_T \rightarrow T$ of π , and for the pullback \mathcal{E}_T of \mathcal{E} to Z_T , the open subset

$U_{\pi_T, \mathcal{E}_T} \subset T$ equals $f^{-1}U_{\pi, \mathcal{E}}$, $(\pi_T)_* \mathcal{E}_T$ is a locally free \mathcal{O}_T -module of (locally) finite rank, and the natural map $f^* \pi_* \mathcal{E} \rightarrow (\pi_T)_* \mathcal{E}_T$ is an isomorphism.

In particular, for a numerical polynomial $P(t)$, for the Hilbert scheme $\text{Hilb}_{\mathbb{P}(E)/S}^{P(t)}$ with its universal closed subscheme $Z \subset \text{Hilb}_{\mathbb{P}(E)/S}^{P(t)} \times_S \mathbb{P}_S(E)$ with its projections

$$\begin{aligned} \pi_{E, P(t)} : Z &\rightarrow \text{Hilb}_{\mathbb{P}(E)/S}^{P(t)}, \\ \rho_{E, P(t)} : Z &\rightarrow \mathbb{P}_S(E) \end{aligned}$$

for every integer $d \geq 1$, there exists a maximal open subscheme $U_d \subset \text{Hilb}_{\mathbb{P}(E)/S}^{P(t)}$ such that $R^q \pi_* \rho^* \mathcal{O}_E(d)$ equals zero on U_d for all $q > 0$. On this open subset, the sheaf $\pi_* \rho^* \mathcal{O}_E(d)$ is locally free. There is a natural base change homomorphism of \mathcal{O}_{U_d} -modules,

$$\phi_d : S_d(E) \otimes_{\mathcal{O}_S} \mathcal{O}_{U_d} \rightarrow \pi_* \rho^* \mathcal{O}_E(d)|_U.$$

Denote by $V_d \subset U_d$ the maximal open subscheme on which ϕ_d is surjective, i.e., V_d is the relative complement in U_d of the support of the cokernel of ϕ_d . In this case, the kernel \mathcal{K}_d of ϕ_d on V_d is locally free. Thus the dual of the kernel, \mathcal{K}_d^\vee , is also locally free on V_d . Denote by $\kappa : \mathbb{P}_{V_d}(\mathcal{K}_d) \rightarrow V_d$ the associated projective bundle with its universal invertible quotient $\kappa^* \mathcal{K}_d^\vee \rightarrow \mathcal{O}_{\mathcal{K}_d}(1)$.

Since $\pi_* \rho^* \mathcal{O}_E(d)|_U$ is locally free on V_d , the associated \mathcal{O}_{V_d} -module homomorphism

$$\psi_d : S_d(E)^\vee \otimes_{\mathcal{O}_S} \mathcal{O}_{V_d} \rightarrow \mathcal{K}_d^\vee$$

is surjective. Thus, there is a unique S -morphism, $\iota : \mathbb{P}_{V_d}(\mathcal{K}_d) \rightarrow \mathbb{P}_S(S_d(E))$, such that $\iota^* \mathcal{O}_{S_d(E)}(1)$ equals $\mathcal{O}_{\mathcal{K}_d}(1)$ and such that ψ_d is the induced homomorphism on global sections of $\mathcal{O}_{S_d(E)}(1)$, resp. $\mathcal{O}_{\mathcal{K}_d}(1)$.

In the special case that the Hilbert polynomial $P(t)$ equals $P_{n,d}(t) = P_n(t) - P_n(t-d)$, this gives the following.

Lemma 2.1. *The closed subscheme $\mathcal{X} \subset \mathbb{P}_S(S_d(E)) \times_S \mathbb{P}_S(E)$ defines an isomorphism from $\mathbb{P}_S(S_d(E))$ to the Hilbert scheme $\text{Hilb}_{\mathbb{P}_S(E)/S}^{P_{n,d}(t)}$.*

Proof. This is well-known. Here is the basic idea. First of all, since \mathcal{X} is a Cartier divisor in a scheme that is flat over $\mathbb{P}_S(S_d(E))$, the Cartier divisor is flat over $\mathbb{P}_S(S_d(E))$ if and only if every geometric fiber is a Cartier divisor in the geometric fiber of $\mathbb{P}_S(E)$. This is true since α is nonzero on geometric fibers. Thus, there is an induced morphism from $\mathbb{P}_S(S_d(E))$ to the Hilbert scheme.

By the computation of cohomology of invertible sheaves on projective space, the image of $\mathbb{P}_S(S_d(E))$ maps into the open subset U_e of the Hilbert scheme for every integer $e \geq d - n$. By computation on geometric points of $\mathbb{P}_S(S_d(E))$, the pullback of ϕ_e is surjective for every $e \geq d - n$. Thus, the morphism to the Hilbert scheme factors through the open subset V_e . On the other hand, on the open subset V_d , since $P_{n,d}(d)$ equals $P_n(d) - 1$, ψ_d is an invertible quotient of the pullback of $S_d(E)^\vee$. This invertible quotient defines an inverse morphism from V_d to $\mathbb{P}_S(S_d(E))$.

Finally, to prove that V_d equals the entire Hilbert scheme, it suffices to compute on geometric points $\text{Spec}(k) \rightarrow S$. For a closed subscheme $Z \subset \mathbb{P}_k(E_k)$ with Hilbert polynomial $P_{n,d}(t)$, since the degree of Hilbert polynomial equals $n - 1$, there are associated primes of Z of height 1, and every such prime is minimal. The intersection of the finitely many primary components of \mathcal{O}_Z for such primes gives

an ideal sheaf whose associated closed scheme $Z^{(1)}$ is a Cartier divisor in $\mathbb{P}_k(E_k)$ contained in Z and that equals the divisorial part of Z . Since the leading coefficient of $P_{n,d}(t)$ equals $d/(n-1)!$, $Z^{(1)}$ has degree d . As a degree d hypersurface in $\mathbb{P}_k(E_k)$, the Hilbert polynomial of $Z^{(1)}$ equals $P_{n,d}(t)$. Thus, for the natural surjection $\mathcal{O}_Z \rightarrow \mathcal{O}_{Z^{(1)}}$, the kernel has Hilbert polynomial zero. Thus the kernel is zero, i.e., Z equals the degree d hypersurface $Z^{(1)}$. \square

Returning to the case of an arbitrary Hilbert polynomial $P(t)$, we have the following generalization.

Proposition 2.2. *Inside $\mathbb{P}_{V_d}(\mathcal{K}_d) \times_S \mathbb{P}_S(E)$, the closed subscheme $(\kappa \times \text{Id}_{\mathbb{P}(E)})^{-1} \mathcal{Z}$ is contained in the closed subscheme $(\iota \times \text{Id})^{-1} \mathcal{X}$. Associated to this pair of closed subschemes, flat over $\mathbb{P}_{V_d}(\mathcal{K}_d)$, the induced morphism from $\mathbb{P}_{V_d}(\mathcal{K}_d)$ to the flag Hilbert scheme $f\text{Hilb}_{\mathbb{P}(E)/S}^{P(t), P_{d,n}(t)}$ is an open immersion whose open image equals the inverse image of V_d via the forgetful morphism $\Phi_{P(t),d} : f\text{Hilb}_{\mathbb{P}(E)/S}^{P(t), P_{d,n}(t)} \rightarrow \text{Hilb}_{\mathbb{P}(E)/S}^{P(t)}$.*

Proof. By construction, on $\mathbb{P}_{V_d}(\mathcal{K}_d)$, the defining polynomials of \mathcal{X} , considered as sections of $S_d(E)$, vanish when restricted to $\pi_* \rho^* \mathcal{O}_E(d)|_{V_d}$. Thus the pullback of \mathcal{Z} is contained in the pullback of \mathcal{X} . Thus, there is an induced morphism to the flag Hilbert scheme. By construction, the image of this morphism is contained in the inverse image of V_d . Now we repeat the argument in the previous lemma to construct an inverse isomorphism from the inverse image of V_d to $\mathbb{P}_{V_d}(\mathcal{K}_d)$. \square

Since \mathcal{K}_d is locally free of rank $P_n(d) - P(d)$, the projection $\mathbb{P}_{V_d}(\mathcal{K}_d) \rightarrow V_d$ is smooth of relative dimension $P_n(d) - P(d)$. Thus, we have a corollary of the previous proposition.

Corollary 2.3. *The forgetful morphism $\Phi_{P(t),d} : \Phi_{P(t),d}^{-1}(V_d) \rightarrow V_d$ is smooth, even a projective bundle, of relative dimension $P_n(d) - P(d)$.*

Using the corollary, the first part of the proposition is reduced to the following result.

Lemma 2.4. *The open subscheme $G_e(r, \mathbb{P}(E))$ of the Hilbert scheme is contained in the open subscheme V_d for every integer $d \geq 1$.*

Proof. Since this is a statement about equality of two open subsets, this can be checked at the level of geometric points of the Hilbert scheme. Thus, assume that k is algebraically closed, and let $\nu : \mathbb{P}_k(E_r) \rightarrow \mathbb{P}_k(E)$ be a Veronese e -uple morphism. For every integer $d \geq 1$, $\nu^* \mathcal{O}_E(d)$ equals $\mathcal{O}_{E_r}(de)$. By the computation of cohomology of line bundles on projective space, $h^q(\mathbb{P}_k(E_r), \mathcal{O}(de))$ is zero for all $q > 0$ and for all $d \geq 1$. Thus, $\text{Image}(\nu)$ gives a point of U_d . Finally, by hypothesis,

$$\nu_1^* : H^0(\mathbb{P}(E)_k, \mathcal{O}_E(1)) \rightarrow H^0(\mathbb{P}(E_r)_k, \mathcal{O}_{E_r}(e)),$$

is surjective. The induced map ϕ_d is just the composite of the d^{th} symmetric power of ν_1^* and the evaluation morphism,

$$\text{Sym}_k^d H^0(\mathbb{P}_k(E), \mathcal{O}_E(1)) \rightarrow \text{Sym}_k^d H^0(\mathbb{P}_k(E_r), \mathcal{O}_{E_r}(e)) \rightarrow H^0(\mathbb{P}_k(E_r), \mathcal{O}_{E_r}(de)).$$

The first factor is surjective by hypothesis, and the second factor is surjective by the computation of cohomology of line bundles on projective space. Thus, $\text{Image}(\nu)$ is a point of V_d . \square

As a special case of the lemma that will be useful later, the $d = 1$ result implies that the following is a short exact sequence of locally free sheaves on $F_e(r, \mathbb{P}(E))$ compatible with arbitrary base change,

$$0 \rightarrow \mathcal{K}_1 \rightarrow E^\vee \otimes_k \mathcal{O}_{G_e(r, \mathbb{P}(E))} \rightarrow \pi_* \rho^* \mathcal{O}_E(1) \rightarrow 0.$$

Denote the quotient by E_G^\vee . Then $\mathbb{P}_G(E_G) \rightarrow G_e(r, \mathbb{P}_k(E))$ is a projective subbundle of the projective bundle $G_e(r, \mathbb{P}_k(E)) \times_{\text{Spec}(k)} \mathbb{P}_k(E)$ that is flat over $G_e(r, \mathbb{P}_k(E))$ of relative dimension $n_e(r)$. By construction, $\mathbb{P}_G(E_G)$ contains the restriction over the open $G_e(r, \mathbb{P}_k(E))$ of the universal closed subscheme over the entire Hilbert scheme. Thus, this pair of closed subschemes gives a morphism to the flag Hilbert scheme,

$$\Psi_{e,r,E} : G_e(r, \mathbb{P}_k(E)) \rightarrow \text{fHilb}_{\mathbb{P}(E)/k}^{P_r(et), P_{n_e(r)}(t)}.$$

On the other hand, there is a forgetful morphism,

$$\Phi : \text{fHilb}_{\mathbb{P}(E)/k}^{P_r(et), P_{n_e(r)}(t)} \rightarrow \text{Hilb}_{\mathbb{P}(E)/k}^{P_r(et)}.$$

By construction, $\Phi \circ \Psi$ is the inclusion, so that the image of Ψ is contained in $\Phi^{-1}(G_e(r, \mathbb{P}_k(E)))$. Altogether, this proves the following.

Corollary 2.5. *The forgetful morphism $\Phi : \Phi^{-1}(G_e(r, \mathbb{P}_k(E))) \rightarrow G_e(r, \mathbb{P}_k(E))$ is an isomorphism, and the pullback via the inverse isomorphism Ψ of the universal linear $n_e(r)$ -fold containing the Veronese is $\mathbb{P}(E_G)$, the family of linear spans of the Veronese e -uple r -folds.*

The last part of Proposition 1.1 follows in characteristic 0 by Generic Smoothness. Of course the characteristic 0 result implies that there exists some integer $p_e(n, r, d)$ such that the result also holds whenever the characteristic p satisfies $p \geq p_e(n, r, d)$. In fact, this integer is effectively computable, even though the effective upper bounds here are probably far from optimal. The key is the following observation.

Lemma 2.6. *Let k be a field. Let S and T be smooth, integral k -schemes. Let $f : S \rightarrow T$ be a dominant morphism. For every irreducible component B of the singular locus of f (defined via Fitting ideals of Ω_f) endowed with its induced reduced structure, if B dominates T , then $B \rightarrow T$ is not separable.*

Proof. Denote by $S^\circ \subset S$, resp. $B^\circ \subset B$, the k -smooth locus of f , resp. of $f|_B$. Then $(df)^\dagger : f^* \Omega_{T/k} \rightarrow \Omega_{S/k}$, resp. $d(f|_B)^\dagger : (f|_B)^* \Omega_{T/k} \rightarrow \Omega_{B/k}$, is a local split injection with locally free cokernel on S° , resp. B° . Since $d(f|_B)^\dagger$ factors through $\Omega_{S/k}|_B \rightarrow \Omega_{B/k}$, it follows that B° is contained in $S^\circ \cap B$. Since B is disjoint from S° , B° is empty. Therefore $f|_B$ is not separable. \square

In case S is a specific quasi-projective T -scheme, up to intersecting S with a sufficiently general collection of hyperplane sections, it suffices to assume that B is generically finite over T . Then, since $B \rightarrow T$ is not separable, the length of $\mathcal{O}_{B,\eta}$ as an $\mathcal{O}_{T,\eta}$ -module, η a generic point of T , is at least p . On the other hand, there

are upper bounds on the length of the singular locus of the zero-dimensional components of the singular locus of S in terms of the dimension and degree of S , cf. [Gut13, Section 4.2]. Using this, it is possible to find an effective upper bound on $p_e(n, r, d)$ in terms of dimensions and degrees of Hilbert schemes.

3. PROOF OF THEOREM 1.3

The proof of the main part of the theorem is very similar to the proof of the theorem of Alex Waldron [Wal08].

Since smoothness can be checked after base change from k to an algebraic closure, assume that k is algebraically closed. As above, assume that E is a k -vector space of rank $n + 1$ so that $(\mathbb{P}_k(E), \mathcal{O}_E(1))$ is k -isomorphic to \mathbb{P}_k^n with its Serre twisting sheaf.

Let E_r be a k -subspace of rank $r + 1$, and let $\nu : \mathbb{P}_k(E_r) \hookrightarrow \mathbb{P}_k(E)$ denote a Veronese e -uple morphism. Denote by \mathcal{J} the corresponding ideal sheaf. In particular, the k -subspace $J_1 := H^0(\mathbb{P}_k(E), \mathcal{J}(1))$ of $E^\vee = H^0(\mathbb{P}_k(E), \mathcal{O}_E(1))$ equals the kernel of the surjection,

$$\nu_1^* : H^0(\mathbb{P}_k(E), \mathcal{O}_E(1)) \rightarrow H^0(\mathbb{P}_k(E_r), \mathcal{O}_{E_r}(e)).$$

This is the same as the pullback of the sheaf \mathcal{K}_1 from the previous section. The annihilator of J_1 is a linear subspace $E_\nu \subset E$ of dimension $P_r(e)$, the pullback of E_G from the previous section. The subvariety $\mathbb{P}_k(E_\nu) = \text{Zero}(J_1)$ of $\mathbb{P}_k(E)$ is the unique linear subvariety of dimension $n_e(r)$ that contains the image of ν , i.e., $\mathbb{P}_k(E_\nu)$ is the linear span of ν . In particular, for the ideal sheaf \mathcal{J}_1 of $\mathbb{P}_k(E_\nu)$ in $\mathbb{P}_k(E)$, $\mathcal{J}_1|_{\mathbb{P}_k(E_\nu)}$ equals $J_1 \otimes_k \mathcal{O}_{E_\nu}(-1)$.

The fundamental exact sequence of sheaves of relative differentials is,

$$0 \longrightarrow \nu^* \mathcal{J} \xrightarrow{\delta} \nu^* \Omega_{\mathbb{P}(E)/k} \xrightarrow{(d\nu)^\dagger} \Omega_{\mathbb{P}(E_r)/k} \longrightarrow 0.$$

Using the Euler exact sequence, $\nu^* \mathcal{J}$ is identified with the locally free sheaf of rank $n - r$, that is the kernel of the associated surjective morphism

$$\widetilde{d\nu}^\dagger : E^\vee \otimes_k \mathcal{O}_{E_r}(-e) \rightarrow E_r^\vee \otimes_k \mathcal{O}_{E_r}(-1).$$

Via the factorization of ν through $\mathbb{P}_k(E_\nu)$, there is an associated short exact sequence for $\nu^* \mathcal{J}$,

$$0 \rightarrow J_1 \otimes_k \mathcal{O}_{E_r}(-e) \rightarrow \nu^* \mathcal{J} \rightarrow \nu^* \mathcal{J}_{>1} \rightarrow 0,$$

where $\mathcal{J}_{>1}$ is the ideal sheaf of $\text{Image}(\nu)$ in $\mathbb{P}_k(E_\nu)$.

Denote by $\Delta_r(e) \subset \mathbb{Z}_{\geq 0}^{r+1}$ the subset of $\underline{e} = (e_0, e_1, \dots, e_r)$ with $e_0 + e_1 + \dots + e_r$ equal to e . This set has size $P_r(e)$. Denote by m the difference $n + 1 - P_r(e)$. Denoting by (t_0, \dots, t_r) a basis for E_r^\vee , and denoting by (y_1, \dots, y_m) a basis for J_1 , this extends to a basis for E^\vee ,

$$(x_{\underline{e}})_{\underline{e} \in \Delta_r(e)} \sqcup (y_1, \dots, y_m),$$

such that for every $\underline{e} = (e_0, e_1, \dots, e_r)$,

$$\nu_1^* x_{\underline{e}} = t_0^{e_0} t_1^{e_1} \dots t_r^{e_r}.$$

Then the restriction k -algebra homomorphism $\nu^* : S(E) \twoheadrightarrow S(E_r)$ is the composition of the quotient $k[x_{\underline{e}}, y_j] \rightarrow k[x_{\underline{e}}]$ by the graded ideal generated by (y_1, \dots, y_m)

and the natural surjection $k[x_e] \rightarrow k[t_0, \dots, t_r]_{(e)}$, where $k[t_0, \dots, t_r]_{(e)}$ is the graded k -subalgebra $\oplus_{d \geq 0} k[t_0, \dots, t_r]_{de}$ of $k[t_0, \dots, t_r]$. The linear space $\text{Zero}(y_1, \dots, y_m)$ equals the linear span of the image of ν , $\text{Span}(\nu)$.

The identity map $S_1(E) \rightarrow E^\vee$ extends uniquely to a k -derivation that also preserves graded decompositions,

$$\partial : S(E) \rightarrow E^\vee \otimes_k S(E)[-1].$$

This k -derivation defines a graded isomorphism of $S(E)$ -modules,

$$\Omega_{S(E)/k} \rightarrow E^\vee \otimes_k S(E)[-1].$$

Similarly, the identity map $S_1(E_r) \rightarrow E_r^\vee$ defines a graded isomorphism of $S(E_r)$ -modules,

$$\Omega_{S(E_r)/k} \rightarrow E_r^\vee \otimes_k S(E_r)[-1].$$

In particular, if e is prime to the characteristic, then the derivation in degree e ,

$$S_e(E_r) \rightarrow E_r^\vee \otimes_k S_{e-1}(E),$$

defines a surjection of $\mathcal{O}_{\mathbb{P}(E_r)}$ -modules,

$$\partial_e : S_e(E_r) \otimes_k \mathcal{O}_{\mathbb{P}(E_r)} \rightarrow E_r^\vee \otimes_k \mathcal{O}_{E_r}(e-1).$$

Twisting, this gives an isomorphism,

$$\nu^* \mathcal{J}_{>1}(e) \cong \text{Ker}(\partial_e).$$

If the characteristic does divide e , then ∂_e has cokernel isomorphic to $\mathcal{O}_{\mathbb{P}(E_r)}(e)$, and then there is a short exact sequence,

$$0 \rightarrow \nu^* \mathcal{J}_{>1}(e) \rightarrow \text{Ker}(\partial_e) \rightarrow \mathcal{O}_{E_r}(e) \rightarrow 0.$$

Lemma 3.1. *Each of $\nu^* \mathcal{J}_1$, $\nu^* \mathcal{J}$, and $\nu^* \mathcal{J}_{>1}$ is a locally free $\mathcal{O}_{\mathbb{P}(E_r)}$ -module. Moreover, for each, h^1 of the dual locally free sheaf is zero.*

Proof. Via the identifications above, it is straightforward to compute that each sheaf is locally free. Moreover, since $\nu^* \mathcal{J}_1^\vee$ is isomorphic to $J_1^\vee \otimes_k \mathcal{O}_{E_r}(e)$, all of the higher cohomology groups of this sheaf are zero. Via the long exact sequence of cohomology, h^1 of $\nu^* \mathcal{J}^\vee$ equals zero if h^1 of $\nu^* \mathcal{J}_{>1}^\vee$ equals zero. Via the isomorphisms and via the vanishing of higher cohomology of $\mathcal{O}_{\mathbb{P}(E_r)}$, h^1 of $\nu^* \mathcal{J}_{>1}^\vee$ equals zero if h^1 of $\text{Ker}(\partial_e)^\vee(e)$ equals zero. If the characteristic is prime to e , resp. divides e , then we have an exact sequence,

$$0 \rightarrow E_r \otimes_k \mathcal{O}_{E_r}(1) \rightarrow S_e(E_r)^\vee \otimes_k \mathcal{O}_{E_r}(e) \rightarrow \text{Ker}(\partial_e)^\vee(e) \rightarrow 0,$$

resp. we have an exact sequence,

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(E_r)} \rightarrow E_r \otimes_k \mathcal{O}_{E_r}(1) \rightarrow S_e(E_r)^\vee \otimes_k \mathcal{O}_{E_r}(e) \rightarrow \text{Ker}(\partial_e)^\vee(e) \rightarrow 0.$$

In each case, using the vanishing of all higher cohomology of $\mathcal{O}_{E_r}(\ell)$ for $\ell > -r$, the long exact sequence of cohomology implies that h^1 of $\text{Ker}(\partial_e)^\vee(e)$ equals 0. \square

For every $G \in S_d(E)$, i.e., for every global section of $\mathcal{O}_E(d)$, the associated adjoint map,

$$\partial G^\dagger : E \rightarrow S_{d-1}(E),$$

sends each dual basis vector of E , x_e^\vee , resp. y_i^\vee to the partial derivative $\partial G / \partial x_e$, resp. $\partial G / \partial y_i$. Assume now that G vanishes on $\text{Span}(\nu)$. (N.B. When $e = 1$, this hypothesis is trivially satisfied. For $e > 1$, one could try to improve the bound in the

theorem by dropping this hypothesis.) Denote by $i : Y \hookrightarrow \mathbb{P}_k(E)$ the zero scheme of G , and denote by \mathcal{I}_Y the corresponding ideal sheaf of $\mathcal{O}_{\mathbb{P}(E)}$. Multiplication by G defines an isomorphism of $\mathcal{O}_{\mathbb{P}(E)}$ -modules, $\mathcal{O}_E(-d) \rightarrow \mathcal{I}_Y$. Thus the fundamental exact sequence of sheaves of relative differentials becomes,

$$0 \longrightarrow i^* \mathcal{O}_E(-d) \xrightarrow{\partial G} i^* \Omega_{\mathbb{P}(E)/k} \xrightarrow{(di)^\dagger} \Omega_{Y/k} \longrightarrow 0.$$

Pulling back to $\mathbb{P}_k(E_\nu)$ and using transitivity for relative differentials, there is a commutative diagram,

$$\begin{array}{ccc} \mathcal{O}_{E_\nu}(-d) & \xrightarrow{\nu^* \partial G} & \Omega_{\mathbb{P}(E)/k}|_{\mathbb{P}(E_\nu)} \\ \partial G_\nu \downarrow & & \downarrow \\ \mathcal{J}_1 & \xrightarrow{\delta} & \Omega_{\mathbb{P}(E_r)/k} \end{array}$$

Via the identification of \mathcal{J}_1 , the homomorphism ∂G_ν is equivalent to a homomorphism,

$$\mathcal{O}_{E_\nu}(-d) \rightarrow (E/E_\nu)^\vee \otimes_k \mathcal{O}_{E_\nu}(-1),$$

Up to a twist and taking the transpose, this is equivalent to a homomorphism,

$$\partial G_{\nu,1}^\dagger : (E/E_\nu) \otimes_k \mathcal{O}_{\mathbb{P}(E_\nu)} \rightarrow \mathcal{O}_{E_\nu}(d-1).$$

Via adjointness of pushforward and pullback, this is equivalent to a homomorphism of k -vector spaces,

$$E/E_\nu \rightarrow S_{d-1}(E_\nu).$$

By abuse of notation, this is also denoted by ∂G_ν^\dagger . This map fits into a commutative diagram,

$$\begin{array}{ccc} E & \xrightarrow{\partial G^\dagger} & S_{d-1}(E) \\ \downarrow & & \downarrow \\ E/E_\nu & \xrightarrow[\partial G_{\nu,1}^\dagger]{} & S_{d-1}(E_\nu) \end{array},$$

where the vertical arrows are the natural surjections. Composing with the surjection ν_e^* , this map induces a k -linear map,

$$G_\nu^\dagger : E/E_\nu \rightarrow S_{(d-1)e}(E_r)$$

For every integer $c \geq 0$, for every integer $b \geq 0$, for every k -vector space W and k -linear map $\phi : W \rightarrow S_b(E)$, there is an associated k -linear map

$$\phi_c : W \otimes_k S_c(E) \rightarrow S_{b+c}(E).$$

obtained from the multiplication on $S(E)$. The k -linear map ϕ is *c-generating* if ϕ_c is surjective, cf. [HS05, Definition 7.2].

Lemma 3.2. *Under the above hypothesis that the degree d hypersurface Y contains the linear span $\mathbb{P}(E_\nu)$ of ν , the smooth locus Y° of the k -scheme Y contains $\text{Image}(\nu)$, resp. $\mathbb{P}(E_\nu)$, if and only if the linear system ∂G_ν^\dagger on $\mathbb{P}(E_r)$, resp. the linear system $\partial G_{\nu,1}^\dagger$ on $\mathbb{P}(E_\nu)$, is c -generating for some $c \geq 1$. When G° contains $\text{Image}(\nu)$, p is smooth at the point corresponding to the pair $([\text{Image}(\nu)], [Y])$ if and only if ∂G_ν^\dagger is e -generating.*

Proof. By the Jacobian criterion, Y^o equals the maximal open subscheme of Y on which $\Omega_{Y/k}$ is locally free of rank $n - 1$. By Nakayama's Lemma, the points of Y are precisely those $\text{Spec } \kappa \rightarrow Y$ where the pullback of ∂G^\dagger is nonzero. By the commutative diagram, this is equivalent to nonvanishing of the pullback of ∂G_j^\dagger . Finally, using the equivalence between the category of coherent sheaves on $\mathbb{P}_k(E_r)$ and the localization of the category of finitely presented, graded $S(E_r)$ -modules with respect to modules concentrated in low degrees, this pullback is nonvanishing at every point of $\mathbb{P}_k(E_r)$, resp. $\mathbb{P}_k(E_\nu)$, if and only if ∂G_ν^\dagger , resp. $\partial G_{\nu,1}^\dagger$ is c -generating for some $c \geq 1$.

Next, assume that ∂G_c^\dagger is c -generating for some $c \geq 1$. Then for the ideal sheaf \mathcal{K} of $\mathbb{P}(E_r)$ in Y , the commutative diagram gives a short exact sequence,

$$0 \longrightarrow \mathcal{O}_{E_r}(-de) \xrightarrow{\partial G_j} \nu^* \mathcal{J} \longrightarrow \nu^* \mathcal{K} \longrightarrow 0$$

Since Y^o contains $\text{Image}(\nu)$, the closed immersion $\nu \mathbb{P}(E_r) \hookrightarrow Y^o$ is a *regular immersion*. Thus the usual obstruction group for deformations of this closed immersion, $\text{Ext}_{\mathcal{O}_Y}^1(\mathcal{K}, \nu_* \mathcal{O}_{\mathbb{P}(E_r)})$ reduces to

$$H^1(\mathbb{P}(E_r), \text{Hom}_{\mathcal{O}_{\mathbb{P}(E_r)}}(\nu^* \mathcal{K}, \mathcal{O}_{\mathbb{P}(E_r)})).$$

Since $\nu^* \mathcal{K}$ is locally free, the transpose of the short exact sequence above is still a short exact sequence. By Lemma 3.1, the long exact sequence defines an isomorphism

$$\delta : \text{Coker}(\partial G_j^\dagger)_1 \xrightarrow{\cong} H^1(\mathbb{P}(E_r), \text{Hom}_{\mathcal{O}_{\mathbb{P}(E_r)}}(j^* \mathcal{K}, \mathcal{O}_{\mathbb{P}(E_r)})).$$

Thus, the obstruction group vanishes if and only if ∂G_ν^\dagger is e -generating.

Of course there are cases where the obstruction group is nonzero, yet the relative Hilbert scheme is still smooth. However, in this case, both the domain and the target of the morphism p are smooth k -schemes. The obstruction group is the cokernel of the map induced by p from the Zariski tangent space of $F_e(r, \mathcal{X}/\mathbb{P}_k(S_d(E)))$ to the Zariski tangent space of $\mathbb{P}_k(S_d(E))$. Thus, by the Jacobian criterion, p is smooth at $([\text{Image}(\nu)], [Y])$ if and only if ∂G_ν^\dagger is e -generating. \square

By Hochster-Laksov [HL87, Theorem 1], for $e = 1$, for all $d \geq 3$, for all $n \geq N_1(r, d) = r + \lceil P_r(d)/(r+1) \rceil$, there exists a linear system of dimension $m = n - r$ in $S_{d-1}(E_r)$ that is 1-generating, say

$$E/E_\nu \rightarrow S_{d-1}(E_r), \quad y_i^\vee \mapsto G_i(t_0, \dots, t_r).$$

Recall that the basis for E^\vee is $(x_0, \dots, x_r) \sqcup (y_1, \dots, y_m)$, where $\nu^* x_i$ equals t_i . For the polynomial

$$G = \sum_{i=1}^m y_i G_i(x_0, \dots, x_r),$$

the zero scheme, Y , of G contains $\mathbb{P}(E_r)$, and ∂G_ν^\dagger is the given 1-generating linear system. Thus, by Lemma 3.2, π is smooth at the pair $([\mathbb{P}(E_r)], [Y])$. This proves Theorem 1.2, and this is basically Waldron's proof. In fact, Waldron also gives a simplified proof of Hochster-Laksov in this case.

Next, for $e \geq 2$, for all $d \geq 3$, it is a theorem of Gleb Nenashev, [Nen16, Theorem 1], that for all integers $m = n - P_r(e)$ satisfying $m \geq P_r(e) + \lceil P_r(de)/P_r(e) \rceil$, there

exists an e -generating linear system,

$$E/E_\nu \rightarrow S_{(d-1)e}(E_r), \quad y_i^\vee \mapsto G_i(t_0, \dots, t_r).$$

For each i , since ν_{d-1}^* is surjective, there exists $H_i \in k[x_\underline{e}]_{d-1}$ such that $\nu^*d - 1(H_i)$ equals G_i . For the polynomial

$$G = \sum_{i=1}^m y_i H_i,$$

the zero scheme, Y , of G contains $\mathbb{P}(E_\nu)$, and ∂G_ν^\dagger is the given e -generating linear system. Thus, by Lemma 3.2, π is smooth at the pair $([\mathbb{P}(E_r)], [Y])$. This proves the Theorem 1.3 for $e \geq 2$ and for $d \geq 3$.

For $d = 1$, for all $n \geq 1 + n_r(e)$, for every hypersurface Y that contains $\text{Image}(\nu)$, $F_e(r, Y) \cong G_e(r, \mathbb{P}_k^{n-1})$ is nonempty and smooth. For $d = 2$ and for $e \geq 2$, there are smooth quadric surfaces that contain $\text{Image}(\nu)$, assuming that k is algebraically closed (it would suffice for k to be infinite). This follows most easily from Bertini's theorem. Since $r \geq 1$, also $2r + 1 \geq 2$. Thus, $P_r(2) \geq 2r + 2$. By Pascal's Theorem, $P_r(t + 1) - P_r(t)$ equals $P_{r-1}(t + 1)$. For $e \geq -r$, resp. for $e \geq -1$, $P_{r-1}(e + 1) \geq 0$, resp. $P_{r-1}(e + 1) > 0$, so that the integer-valued function $P_r(e)$ is nondecreasing, resp. increasing, in e for $e \geq -r$, resp. $e \geq -1$. Thus, for all $e \geq 2$, $P_r(e) \geq P_r(2) \geq 2r + 2$. Thus, for $n \geq n_r(e) = P_r(e) - 1$, n is strictly larger than $2r$, $P_r(e) - 1 \geq 1 + 2r$. Thus, by the usual parameter counting proof of Bertini's theorem, to prove that a general member G in $H^0(\mathbb{P}_k(E), \mathcal{J}(2))$ defines an everywhere smooth quadric, it suffices to prove for every k -point $p \in \mathbb{P}_k(E_r)$ that the induced map,

$$H^0(\mathbb{P}_k(E), \mathcal{J}(2)) \rightarrow T_{\nu(p)}\mathbb{P}_k(E)/d\nu(T_p\mathbb{P}_k(E_r)),$$

is surjective.

Choose homogeneous coordinates on $\mathbb{P}_k(E_r)$ so that p equals $[t_0, t_1, \dots, t_r] = [1, 0, \dots, 0]$, and then choose corresponding homogeneous coordinates $(x_\underline{e}, y_i)$ on $\mathbb{P}_k(E)$ as above. Then $\nu(p)$ is the point where the coordinate $x_{(e, 0, \dots, 0)} \neq 0$, yet $x_\underline{e} = 0$ for every $\underline{e} \in \Delta_r(e) \setminus \{(e, 0, \dots, 0)\}$, and $y_i = 0$ for every $i = 1, \dots, m$.

The tangent space of $T_p\mathbb{P}_k(E_r)$ is the space spanned by the partial derivatives $\partial/\partial(x_\underline{e}/x_{(e, 0, \dots, 0)})$ for the elements $\underline{e} = (e - 1, 0, \dots, 0, 1, 0, \dots, 0)$. The quotient space is generated by the partial derivatives for $y_i/x_{(e, 0, \dots, 0)}$ for $i = 1, \dots, m$, and by the partial derivatives of $x_\underline{e}/x_{(e, 0, \dots, 0)}$, where $\underline{e} = (e_0, e_1, \dots, e_r)$ satisfies $e_0 \leq e - 2$. For every $i = 1, \dots, m$, the quadratic polynomial $y_i x_{(e, 0, \dots, 0)}$ maps to the image of the partial derivative for $y_i/x_{(e, 0, \dots, 0)}$. For every $\underline{e} \in \Delta_r(e)$ with $e_0 \leq e - 2$, there exist elements $\underline{e}', \underline{e}'' \in \Delta_r(e)$ with $e'_0, e''_0 \leq e - 1$ such that $\underline{e} + (e, 0, \dots, 0) = \underline{e}' + \underline{e}''$. Thus the quadratic polynomial $x_\underline{e} x_{(e, 0, \dots, 0)} - x_{\underline{e}'} x_{\underline{e}''}$ maps to the image of the partial derivative for $x_\underline{e}/x_{(e, 0, \dots, 0)}$. Thus, by Bertini's Theorem, there exists $G \in H^0(\mathbb{P}_k(E), \mathcal{J}(2))$ such that $Y = \text{Zero}(G)$ is everywhere smooth. The action of $\mathbf{PGL}(E)$ on the open subset $\mathbb{P}S_2(E) \setminus \Delta$ parameterizing smooth quadrics is smoothly homogeneous. This action lifts to an action of $\mathbf{PGL}(E)$ on $F_e(r, \mathcal{X})$. Thus, whenever Y is smooth, the restriction of π to the $\mathbf{PGL}(E)$ -orbit of $([\text{Image}(\nu)], [Y])$ is smooth, cf. [Gro67, Corollaire 6.5.2(i)].

By the lemma, if π is smooth, then $n - r \geq n_0 - r$. Conversely, assume that $n \geq n_0$. Then there exists a k -linear map,

$$\phi : E/E_r \rightarrow S_{d-1}(E_r)$$

that is 1-generating. For the image of every dual vector t_i^\vee in E/E_r , denote by $G_i \in S_{d-1}(E_r)$ the image of this element under ϕ . For every $(n-r)$ -tuple $(\tilde{G}_{r+1}, \dots, \tilde{G}_n)$ of elements $\tilde{G}_i \in S_{d-1}(E)$ that maps to G_i , the element

$$G = \hat{G} + t_{m+1}\tilde{G}_1 + \dots + t_n\tilde{G}_n$$

is an element of $S_d(E)$ that vanishes on $\mathbb{P}(E_r)$. By definition of ∂G_j^\dagger in terms of partial derivatives, this equals ϕ . Thus, $([\mathbb{P}(E_r)], [Y])$ is a point where p is smooth. This proves the first part of the proposition: the smooth locus of p is nonempty (and hence dense) if and only if $n \geq n_0$.

4. PROOF OF PROPOSITION 1.4

In this section, fix e to equal 1. Let n equal $n'_1(d, r)$, and denote $m = n'_1(d, r) - r$. The morphism $\rho : F_e(r, \mathcal{X}) \rightarrow G_e(r, \mathbb{P}(E))$ is a Zariski locally trivial projective bundle. Moreover, both domain and target have natural actions of $\mathbf{PGL}(E)$, and the morphism is equivariant for these actions. Finally, the morphism π is also equivariant. Thus the closed subscheme B where π is not smooth is $\mathbf{PGL}(E)$ -invariant. Since $G_e(r, \mathbb{P}(E))$ is homogeneous under the action of $\mathbf{PGL}(E)$, the restriction of ρ to B is flat. Thus, the hypothesis that B has codimension ≥ 2 everywhere is equivalent to the hypothesis that the intersection of B with one, and hence every, geometric fiber of ρ has codimension ≥ 2 everywhere in that fiber. Since the geometric fibers of ρ are projective spaces, this is equivalent to the hypothesis that there exists a finite morphism from \mathbb{P}^1 to a geometric fiber of ρ whose image is disjoint from B .

Now we apply the proof of Bertini's Connectedness Theorem, as generalized by Cristian Minoccheri. For the k -morphism,

$$\pi : F_1(r, \mathcal{X}) \rightarrow \mathbb{P}_k(S_d(E)),$$

the source and target are both smooth, projective k -schemes, and the target is algebraically simply connected, since it is a projective space. By hypothesis, the complement B of the smooth locus of π has codimension ≥ 2 everywhere. Thus, by [Min16, Theorem 3.1], the geometric generic fiber of π is connected. Finally, by Zariski's Main Theorem, since the geometric generic fiber of π is connected, every geometric fiber of π is connected.

For the maximal open subscheme U of $\mathbb{P}_k(S_d(E))$ over which both $F_1(r, \mathcal{X})$ and B are flat, the restriction of π over U is a flat morphism whose domain and target are both smooth, hence $\pi|_U$ is a flat, local complete intersection morphism [Ful84, Appendix B.7.6]. So every geometric fiber of π over U is a projective, local complete intersection scheme. Moreover, the singular locus equals the intersection of the fiber with B , and this has codimension ≥ 2 by hypothesis. Thus, by Serre's Criterion, the geometric fiber is integral and normal, cf. [Gro67, Théorème 5.8.6]. As in Proposition 1.1, if the characteristic equals 0 or $p \geq p_e(n, r, d)$, then, up to replacing U by a dense, Zariski open subscheme, $\pi|_U$ is even smooth.

5. PROOF OF COROLLARY 1.5

Recall from Corollary 2.5 that there exists a universal family of linear spans $\mathbb{P}_G(E_G) \subset G_e(r, \mathbb{P}_k(E)) \times_{\text{Spec}(k)} \mathbb{P}_k(E)$ of the universal family of Veronese varieties. This projective subbundle contains the universal family of Veronese e -uple r -folds. The pair defines a morphism to the flag Hilbert scheme,

$$\Psi_{e,r,E} : G_e(r, \mathbb{P}_k(E)) \rightarrow \text{fHilb}_{\mathbb{P}(E)/k}^{P_r(et), P_{n_e(r)}(t)},$$

whose image is contained in the inverse image open subset $\Phi^{-1}(G_e(r, \mathbb{P}_k(E)))$, where Φ is the forgetful morphism

$$\Phi : \text{fHilb}_{\mathbb{P}(E)/k}^{P_r(et), P_{n_e(r)}(t)} \rightarrow \text{Hilb}_{\mathbb{P}(E)/k}^{P_r(et)}.$$

Now consider the second forgetful morphism,

$$\Lambda : \Phi^{-1}(G_e(r, \mathbb{P}_k(E))) \rightarrow G_1(n_e(r), \mathbb{P}_k(E)).$$

Using the action of $\mathbf{PGL}(E)$, the morphism Λ is a Zariski locally trivial fiber bundle whose fiber over a κ -valued point $[\mathbb{P}_\kappa(E')] \in G_1(n_e(r), \mathbb{P}_k(E))(\text{Spec}(\kappa))$ equals $G_e(r, \mathbb{P}_\kappa(E'))$. In particular, Λ is faithfully flat, finitely presented, quasi-projective and smooth with geometrically irreducible fibers.

Now let $d \geq 1$ be an integer. As usual, denote by $\mathcal{X} \subset \mathbb{P}H^0(\mathbb{P}_k(E), \mathcal{O}_E(d)) \times_{\text{Spec}(k)} \mathbb{P}_k(E)$ the universal family of degree d hypersurfaces in $\mathbb{P}_k(E)$. Consider the projection

$$\rho : F_1(n_e(r), \mathcal{X}) \rightarrow G_1(n_e(r), \mathbb{P}_k(E)).$$

Denote by $F_{e,1}(r, n_e(r), \mathcal{X})$ the fiber product,

$$\begin{array}{ccc} F_{e,1}(r, n_e(r), \mathcal{X}) & \xrightarrow{\text{pr}_1} & \Phi^{-1}(G_e(r, \mathbb{P}_k(E))) \\ \text{pr}_2 \downarrow & & \downarrow \Lambda \\ F_1(n_e(r), \mathcal{X}) & \xrightarrow{\rho} & G_1(n_e(r), \mathbb{P}_k(E)). \end{array}$$

Chasing diagrams, $F_{e,1}(r, n_e(r), \mathcal{X})$ is an open subset of the relative flag Hilbert scheme of $\pi : \mathcal{X} \rightarrow \mathbb{P}H^0(\mathbb{P}_k(E), \mathcal{O}_E(d))$ parameterizing pairs of closed subschemes in fibers of π of Hilbert polynomials $P_r(et)$, resp. $P_{n_e(r)}(t)$. More precisely, this is the open subset of the flag Hilbert scheme parameterizing pairs where the smaller closed subscheme is a Veronese e -uple r -fold, and where the larger closed subscheme is the linear span of the Veronese variety. In particular, because Λ is faithfully flat, finitely presented, quasi-projective and smooth with geometrically irreducible fibers, the same holds for the base change morphism,

$$\text{pr}_2 : F_{e,1}(r, n_e(r), \mathcal{X}) \rightarrow F_1(n_e(r), \mathcal{X}).$$

Since $n \geq n'_1(d, n_e(r))$, the projection morphism

$$\pi' : F_1(n_e(r), \mathcal{X}) \rightarrow \mathbb{P}H^0(\mathbb{P}_k(E), \mathcal{O}_E(d))$$

is projective and dominant with irreducible geometric generic fiber, by Proposition 1.4. Combined with the previous paragraph, also the composition

$$F_{e,1}(r, n_e(r), \mathcal{X}) \xrightarrow{\text{pr}_2} F_1(n_e(r), \mathcal{X}) \xrightarrow{\pi'} \mathbb{P}H^0(\mathbb{P}_k(E), \mathcal{O}_E(d)),$$

is quasi-projective and dominant with irreducible geometric generic fiber. By the definition of the flag Hilbert scheme, this composition also equals the composition

$$F_{e,1}(r, n_e(r), \mathcal{X}) \xrightarrow{\text{pr}_1} F_e(r, \mathcal{X}) \xrightarrow{\pi} \mathbb{P}H^0(\mathbb{P}_k(E), \mathcal{O}_E(d)).$$

By Proposition 1.1, $F_e(r, \mathcal{X})$ is smooth and irreducible, even a projective bundle over $G_e(r, \mathbb{P}_k(E))$. In particular, the image of pr_1 is contained in the normal locus of $F_e(r, \mathcal{X})$. Thus, by [dJS06, Lemma 3.2], also the morphism

$$\pi : F_e(r, \mathcal{X}) \rightarrow \mathbb{P}H^0(\mathbb{P}_k(E), \mathcal{O}_E(d))$$

is dominant with irreducible geometric generic fiber. By the usual constructibility argument, cf. [Jou83, Théorème I.4.10], there exists a dense open subset $W_{e,d}^r$ of $\mathbb{P}H^0(\mathbb{P}_k(E), \mathcal{O}_E(d))$ over which π is faithfully flat with geometrically irreducible fibers.

6. PROOF OF THEOREM 1.6

Proof of (i). Since $d \geq 2$, also $d - 1 \geq 1$. Thus the difference $m = n - r - 1$ satisfies $m \geq n_1(d, r) - r$, and this is at least $r + 1$. In particular, m is positive.

Choose homogeneous coordinates $(x_0, \dots, x_r, y_0, y_1, \dots, y_m)$ on $\mathbb{P}_k(E)$ so that $\mathbb{P}_k(E_r)$ equals the zero scheme of (y_0, y_1, \dots, y_m) . Assume that $n \geq 1 + n_1(r)$. Then by Hochster-Laksov once again, there exists a 1-generating k -subspace $\iota : W \hookrightarrow k[t_0, \dots, t_r]_{d-1}$ of dimension m . Let (w_1, \dots, w_m) be an ordered basis for W , and denote by $G_i(t_0, \dots, t_r)$ the image $\iota(w_i)$. Define $W' = k^{\oplus(m+1)}$, and define

$$\psi : (k^{\oplus 2}) \times (k^{\oplus(m+1)}) \rightarrow W,$$

$$\psi((a, b), (c_0, \dots, c_m)) = (ac_0 + bc_1)w_1 + (ac_1 + bc_2)w_2 + \dots + (ac_{m-1} + bc_m)w_m.$$

This is a k -bilinear map. When a is nonzero, then the restriction of $\psi_{(a,b), \bullet}$ to the subspace $\text{Zero}(c_m)$ is an isomorphism, so that the image is 1-generating. When b is nonzero, then the restriction of $\psi_{(a,b), \bullet}$ to the subspace $\text{Zero}(c_0)$ is an isomorphism. Thus, defining

$$G_{a,b}(x_i, y_j) = (ay_0 + by_1)G_1(x_0, \dots, x_r) + \dots + (ay_{m-1} + by_m)G_m(x_0, \dots, x_r),$$

and defining $Y_{a,b} = \text{Zero}(G_{a,b}) \subset \mathbb{P}_k(E)$, there is a morphism

$$g : \mathbb{P}_k^1 \rightarrow F_1(r, \mathcal{X}), \quad [a, b] \mapsto ([\mathbb{P}_k(E_r)], [Y_{a,b}]),$$

that is a finite morphism into the fiber of ρ over $[\mathbb{P}_k(E_r)]$ whose image is contained in the smooth locus $F_e(r, \mathcal{X})_{\text{sm}}$ of π , i.e., the image of g is disjoint from the singular locus B of π . The fiber of ρ is a projective space, and every nonempty Cartier divisor in projective space has nonempty intersection with every nonempty curve in projective space. Thus, the intersection of B with the fiber of ρ has codimension ≥ 2 in that fiber. As in the proof of Proposition 1.4, since both ρ and $\rho|_B : B \rightarrow G_1(r, \mathbb{P}_k(E))$ are flat, it follows that B has codimension ≥ 2 everywhere in $F_1(r, \mathcal{X})$. Thus, $n'_1(d, r)$ is no greater than $1 + n_0(d, r)$.

Proof of (ii). This is essentially the same argument as in the proof of Corollary 1.5. The morphism $\Lambda : F_e(r, \mathcal{X}) \rightarrow \mathbb{P}H^0(\mathbb{P}_k(E), \mathcal{O}_E(1))$ is a Zariski locally trivial fiber bundle whose fibers are schemes $G_e(r, \mathbb{P}_k^{n-1})$. These are nonempty and geometrically connected precisely when $n - 1 \geq n_e(r)$, i.e., $n \geq 1 + n_e(r)$.

Proof of (iii). By (i), $n'_1(2, r) \leq 1 + n_1(2, r)$. Thus, it suffices to prove that $F_1(r, X)$ is disconnected for $n = n_1(2, r) = 2r + 1$. Denote by $(t_0, \dots, t_r, t_{r+1}, \dots, t_{2r+1})$ an

ordered basis for $S_1(E)$. For k algebraically closed, all smooth quadric hypersurfaces in $\mathbb{P}_k(E)$ are projectively equivalent to the zero scheme of the quadratic polynomial,

$$G(t_0, t_1, \dots, t_r, t_{r+1}, t_{r+2}, \dots, t_{2r+1}) = t_0 t_{r+1} + t_1 t_{r+2} + \dots + t_r t_{2r+1}.$$

Consider the r -planes $\Pi = \text{Zero}(t_{r+1}, \dots, t_{2r+1})$, $\Lambda = \text{Zero}(t_0, \dots, t_r)$, and $\Gamma = \text{Zero}(t_0, t_{r+2}, t_{r+3}, \dots, t_{2r+1})$. These are all contained in $X = \text{Zero}(G)$. By [BHB06, Lemma 0.3, Appendix], $(\Pi.\Lambda)_X = 0$. If m is even, then $(\Pi.\Pi)_X = 1$. If m is odd, then $(\Pi.\Gamma)_X = 1$. Since algebraically equivalent m -cycles are numerically equivalent, it follows that $F_m(X/k)$ has more than one connected component (in fact it has precisely two connected components).

Proof of (iv). This is a computation in the Chow group of the Grassmannian $G_1(r, \mathbb{P}_k(E))$. As an Abelian group under addition, this is a finite free Abelian group. Moreover, for the projection of the flag variety to the Grassmannian,

$$\text{Flag}(0, 1, \dots, r, \mathbb{P}_k(E)) \rightarrow G_1(r, \mathbb{P}_k(E)),$$

the induced pullback map on Chow rings is an injective homomorphism that identifies the Chow ring of $G_1(r, \mathbb{P}_k(E))$ with a saturated Abelian subgroup of the Chow ring of the flag variety, i.e., the quotient Abelian group is a finite free Abelian group. Thus, divisibility of cycles in the Chow ring of $G_1(r, \mathbb{P}_k(E))$ can be checked after pullback to the Chow ring of the flag variety.

There are many methods for performing computations in the Chow ring of the flag variety. The method used here is via “Chern roots” of the total Chern class of the tautological bundle. Begin with the $(r+1)$ -fold fiber product,

$$\mathbb{P}_k(E)^{r+1} = \mathbb{P}_k(E) \times_{SP(k)} \dots \times_{\text{Spec}(k)} \mathbb{P}_k(E).$$

Inside of this scheme, denote by $D_{\leq r}$ the degeneracy closed subscheme (in the sense of Porteous’s formula) where the $r+1$ points are linearly degenerate, i.e., the closed subscheme defined by the vanishing of all $(r+1) \times (r+1)$ -minors of the following homomorphism of locally free sheaves,

$$\phi_{r+1} : E^\vee \otimes_k \mathcal{O}_{\mathbb{P}(E)^{r+1}} \rightarrow \bigoplus_{i=0}^r \text{pr}_i^* \mathcal{O}_{\mathbb{P}(E)}(1).$$

Denote by $U_{r+1} \subset \mathbb{P}_k(E)^{r+1}$ the open complement of $D_{\leq r}$. On U_{r+1} , the morphism ϕ_{r+1} is a locally free quotient of rank $r+1$. Thus, for every integer $0 \leq s \leq r$, the associated map

$$\phi_{r+1, s+1} : E^\vee \otimes_k \mathcal{O}_{U_{r+1}} \rightarrow \bigoplus_{i=0}^s \text{pr}_i^* \mathcal{O}_{\mathbb{P}(E)}(1)$$

is a locally free quotient of rank $s+1$. Altogether, these morphisms define a morphism to the flag variety,

$$\beta : U_{r+1} \rightarrow \text{Flag}(0, 1, \dots, r, \mathbb{P}_k(E)).$$

Working inductively on r , β is an iterated fiber bundle, each factor of which is an affine space bundle that is trivialized for a Zariski open covering of the target. Thus, by the homotopy axiom for Chow groups, [Ful84, Proposition 1.9], the pullback map

$$\text{CH}^*(\text{Flag}(0, 1, \dots, r, \mathbb{P}_k(E))) \rightarrow \text{CH}^*(U_{r+1})$$

is a ring isomorphism.

On the other hand, since U_{r+1} is an open subset of $\mathbb{P}_k(E)^{r+1}$, there is a presentation for the Chow group,

$$\mathrm{CH}^*(\mathbb{P}_k(E)^{r+1})/I \cong \mathrm{CH}^*(U_{r+1}),$$

where I is a \mathfrak{S}_{r+1} -invariant ideal. Since $\mathrm{CH}^*(\mathbb{P}_k(E)) = \mathbb{Z}[u]/\langle u^{n+1} \rangle$ for the first Chern class u of $\mathcal{O}_E(1)$, this presentation is the same as as

$$\mathbb{Z}[u_0, \dots, u_r]/J$$

where J is an ideal containing $\langle u_0^{n+1}, \dots, u_r^{n+1} \rangle$ such that $J/\langle u_0^{n+1}, \dots, u_r^{n+1} \rangle$ equals I , and each u_i is the first Chern class of $\mathrm{pr}_i^* \mathcal{O}_{\mathbb{P}(E)}(1)$. Moreover, for every integer $s = 0, \dots, r$, for the tautological locally free quotient bundle $E^\vee \otimes_k \mathcal{O}_{\mathrm{Flag}} \rightarrow Q_{s+1}$ of rank $s+1$, the total Chern class of Q_{s+1} equals the image of $(1+u_0)(1+u_1)\cdots(1+u_s)$. Thus, the elements (u_0, \dots, u_r) are the ‘‘Chern roots’’ of the tautological flag of locally free sheaves on the flag variety (up to signs, depending on the sign convention; these signs have no effect on divisibility).

In particular, the top Chern class of $\mathrm{Sym}^d(Q_{r+1})$ equals the image of the \mathfrak{S}_{r+1} -invariant polynomial,

$$p_{r+1,d}(u_0, \dots, u_r) = \prod_{\underline{d}, d_0+\dots+d_r=d} (d_0 u_0 + \dots + d_r u_r),$$

where the product is over all elements $\underline{d} = (d_0, \dots, d_r)$ in $(\mathbb{Z}_{\geq 0})^{r+1}$ with $d_0 + \dots + d_r = d$. In particular, separating out those factors $\underline{d} = d\mathbf{e}_i$ where only $d_i = d$ and all other d_j are zero, $p_{r+1,d}$ factors as

$$\begin{aligned} p_{r+1,d}(u_0, \dots, u_r) &= d^{r+1}(u_0 u_1 \cdots u_r) q_{r+1,d}(u_0, \dots, u_r), \\ q_{r+1,d}(u_0, \dots, u_r) &= \prod_{\underline{d} \neq d\mathbf{e}_i, d_0+\dots+d_r=d} (d_0 u_0 + \dots + d_r u_r). \end{aligned}$$

Thus, the top Chern class of $\mathrm{Sym}^d(S_{r+1}^\vee)$ equals d^{r+1} times another class, in fact $d^{r+1} c_{r+1}(S_{r+1}^\vee) \gamma$ where γ is the class obtained as the image of $q_{r+1,d}(u_0, \dots, u_r)$.

A priori, this top Chern class might be zero as a cycle class. However, by Theorem 1.2, when $f_1(n, r, d) \geq 0$, this class is nonzero: it is Poincaré dual to the transversal cycle $F_1(r, X)$ for sufficiently general X . Moreover, since $F_1(r, X)$ is generically smooth for general X , in the special case that $f_1(n, r, d)$ equals 0, $F_1(r, X)$ is a zero-dimensional, smooth k -scheme whose length equals the degree of this top Chern class. By the computation above, the degree of the top Chern class in $\mathrm{CH}^{(r+1)(n-r)}(G_1(r, \mathbb{P}_k(E)))$ equals d^{r+1} times the degree of another cycle. Thus the length of the zero-dimensional, smooth k -scheme $F_1(r, X)$ is divisible by d^{r+1} . In particular, for $d \geq 2$, $F_1(r, X)$ is not geometrically connected as a k -scheme.

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